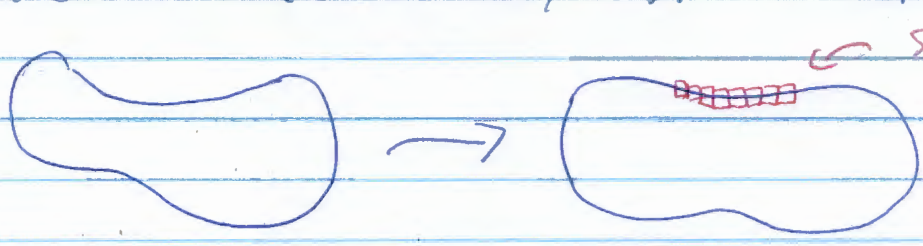
Measure Zero & Osgood's Construction.

Remark on Tutorial 2:

- To prove that every continuous function is integrable over compact set, we use the following fact:

(*): $\forall \varepsilon > 0$, $\exists$ a finite collection of rectangle $\{R_i\}_{i=1}^n$ covering the curve ∂D (boundary of D) s.t. $\sum_{i=1}^n \text{Area}(R_i) < \varepsilon$.



sum of area is very small so that it contributes nothing to the integral.

Defn: (Measure Zero)

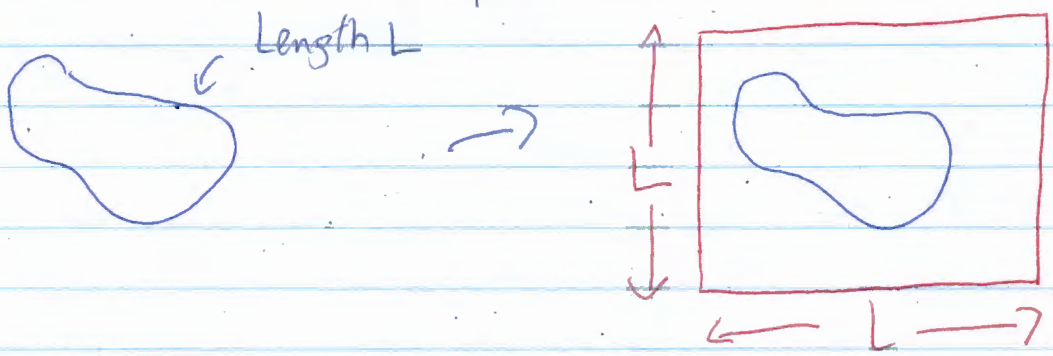
- A curve $c: [0,1] \to \mathbb{R}^2$ is <u>measure zero</u> if it satisfies (*).

Intuitively, the curve ∂D should have finite length, here we give a rigorous proof for this case:

Prop: Every curve with finite length is measure zero.

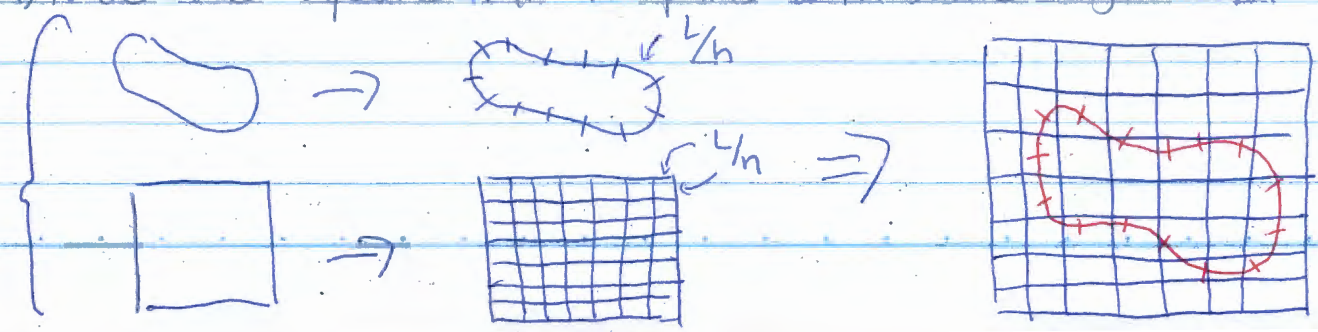
Pf: Let L be the length of the curve c.

Put the curve inside a square with length L.

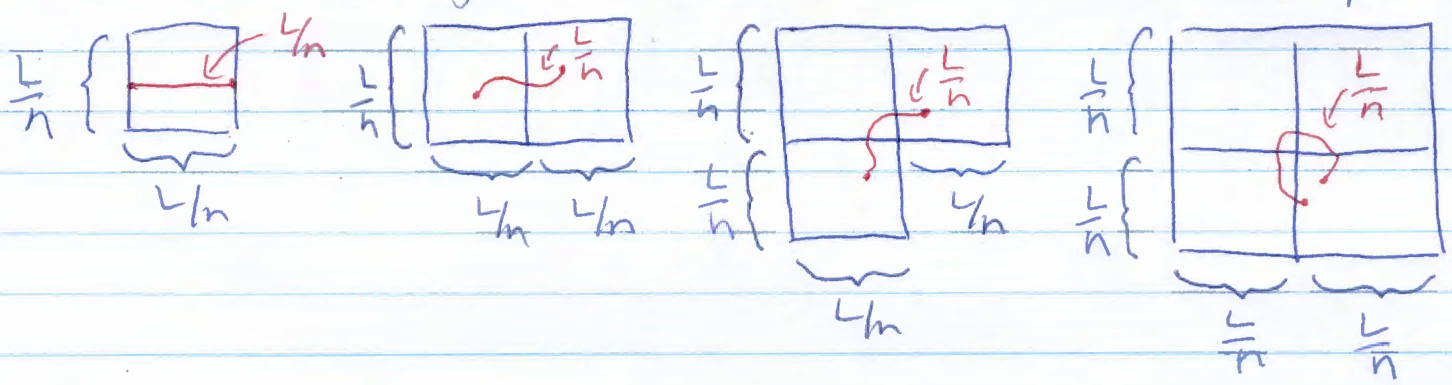


Divide c into n parts with same length L/n.

Divide the square into n^2 parts with same length L/n.



Note that each segments of the curves lies in at most 4 square:



Therefore, the sum of area of square covering the curve c

$$\leq \underbrace{n}_{\text{no. of segments of length } \frac{L}{n}} \cdot \underbrace{4}_{\text{max. no. of square covering a segment}} \cdot \underbrace{\left(\frac{L}{n}\right)^2}_{\text{area of each square covering a segment.}}$$

$$= \frac{4L^2}{n} \to 0 \quad \text{as} \quad n \to \infty \qquad \blacksquare$$

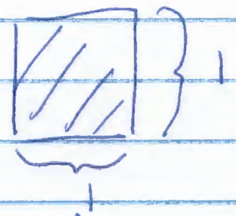
However, if the curve c has infinite length (although it's counter-intuitive), the argument does not work.
Actually, mathematician Osgood (1864–1943) constructed a curve that has non-zero measure.
Now we are going to construct such a curve.

Remark: (1) There are many counter-intuitive phenomenon in maths, such as Weierstrass function (continuous but nowhere differentiable function) and space-filling curve (a surjective map from $[0,1]$ to $[0,1]^2$).
Osgood's curve is also an example for such phenomenon.
(2) The following argument is not easy for Year 1/2 undergraduate. It will be better if you already know what is Cantor set. Please feel free to skip the mathematical part if it is difficult for you.

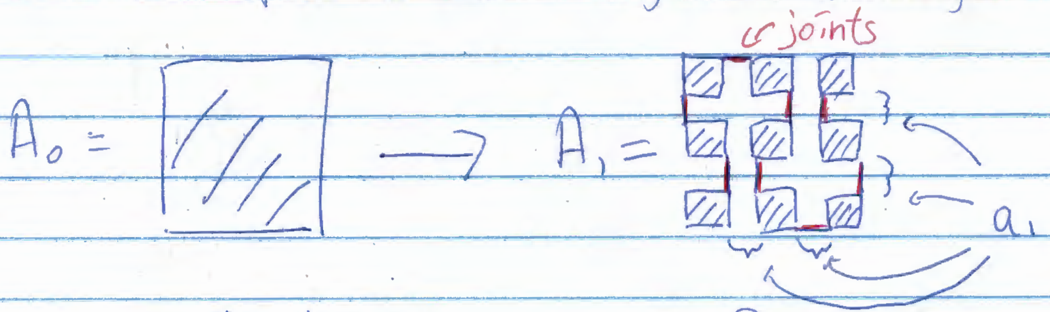
(3) Every thing below is out of syllabus and is just for your interest.

Osgood's Construction:

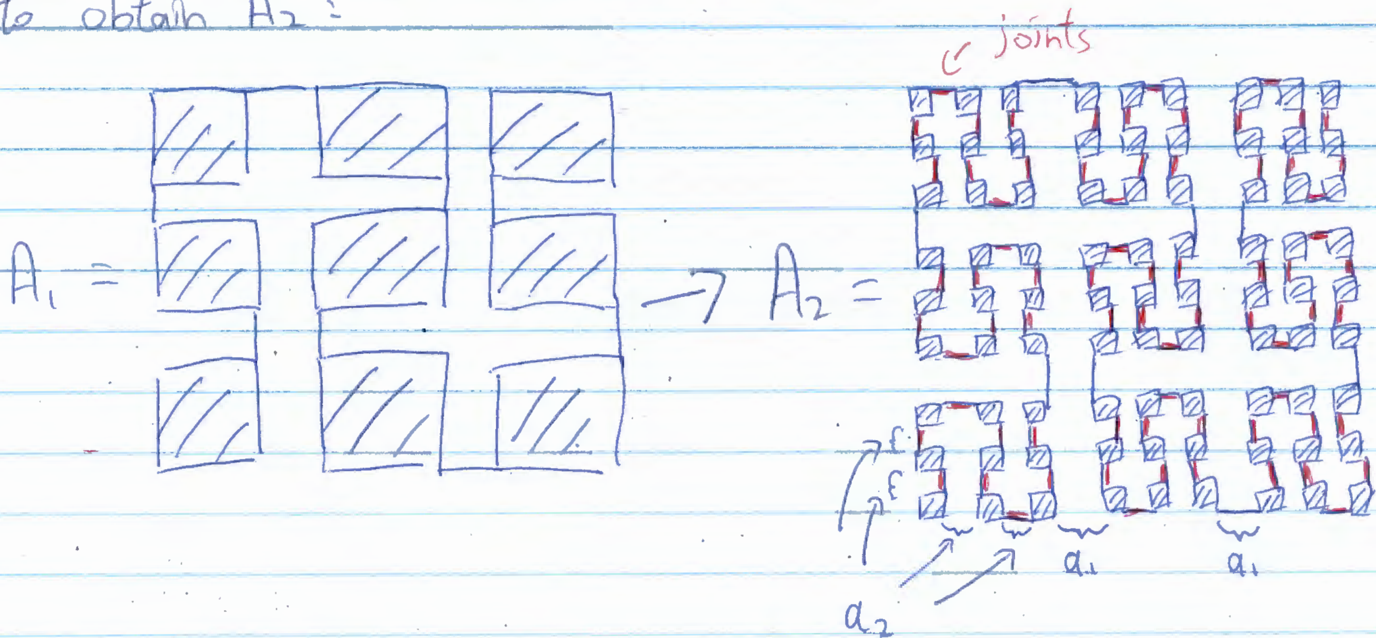
Take a unit square on a plane: $A_0 =$



Then we do the operation below to get a new figure:



Here we get 9 identical squares with 8 joints.
On each smaller square, we do the operation again to obtain A_2:



Here we get 9^2 identical squares with $8 \cdot 9$ joints.

Recursively, we get a sequence of region $A_0 \supset A_1 \supset A_2 \supset A_3 \supset \cdots \supset A_n \supset \cdots$
and $1 > a_1 > a_2 > a_3 > \cdots > a_n > \cdots$

In particular, choose $a_n = \frac{1}{6^n}$.

Then area of $A_1 = \underset{\text{no. of square}}{9} \cdot \underbrace{\left(\underset{\text{length of each square}}{\frac{1-2\left(\frac{1}{6}\right)}{3}}\right)^2}_{\text{area of square}}$

$$\text{area of } A_2 = 9^2 \left(\frac{\frac{1-2\left(\frac{1}{6}\right)}{3} - 2\left(\frac{1}{6^2}\right)}{3} \right)^2$$

$$\vdots$$

$$\text{area of } A_n = 9^n \left(\frac{\frac{\frac{1-2\left(\frac{1}{6}\right)}{3} - 2\left(\frac{1}{6^2}\right)}{3} - 2\left(\frac{1}{6^3}\right)}{3} - \cdots - 2\left(\frac{1}{6^n}\right) \over 3 \right)^2$$

$$= 9^n \left(\frac{1}{3^n} - \frac{2}{3^n}\left(\frac{1}{6}\right) - \frac{2}{3^{n-1}}\left(\frac{1}{6^2}\right) - \cdots - \frac{2}{3}\left(\frac{1}{6^n}\right) \right)^2$$

$$= 9^n \left(\frac{1}{3^n} - \frac{1}{3^{n+1}} \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \right) \right)^2$$

$$= \left(1 - \frac{2}{3} \frac{\left(\frac{1}{2}\right)\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} \right)^2$$

$$= \left(1-\frac{2}{3}\left(1-\left(\frac{1}{2}\right)^n\right)\right)^2$$

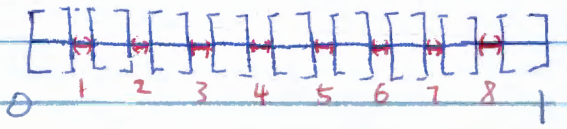
$$\to \frac{1}{9} \quad \text{as} \quad n \to \infty.$$

Hence, we finally construct a region $A_\infty = \bigcap_{i=1}^{\infty} A_i$ with area $\frac{1}{9}$.

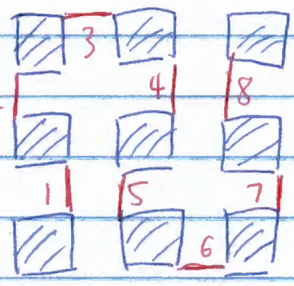
Next, we are going to show that the region A_∞ is actually a curve, i.e. we can parametrize A_∞ by $[0,1]$ continuously and injectively.

We are going to parametrize the joints of A_n one by one.

For A_1, we parametrize the joints linearly in this way:



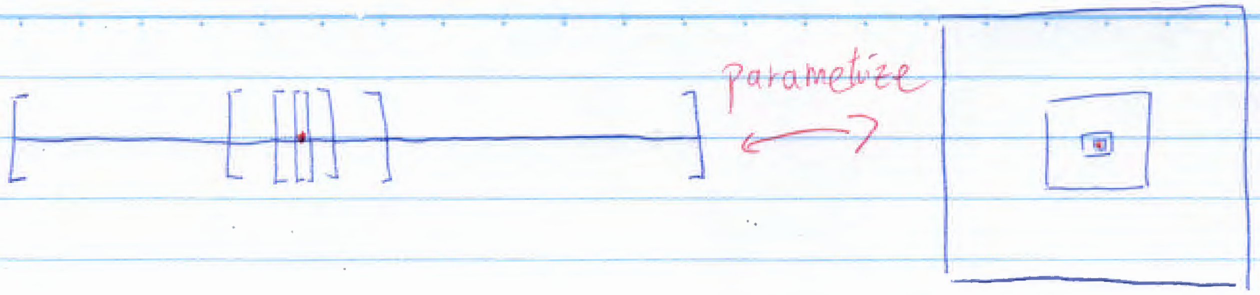
$A_1 =$



Divide $[0,1]$ into 9 subinterval $[0,\frac{1}{17}], [\frac{2}{17},\frac{3}{17}], \dots, [\frac{16}{17},1]$ and leave with 8 open interval $(\frac{1}{17},\frac{2}{17}), (\frac{3}{17},\frac{4}{17}), \dots, (\frac{15}{17},\frac{16}{17})$.
Parametrize the joints in the obvious way.

Recursively, do the parametrization in the same way for each of the closed interval obtained from previous step.
We parametrize all the joints. Clearly the parametrization is continuous and injective.

Finally, we are going to parametrize the points which are not lie on any joints.
In other word, it must lie on squares in each step of the construction.



As the size of the intervals and squares go to 0, $\exists$ unique point lies in all intervals and $\exists$ unique point lies in all square.
We parametize the point as shown above.
Continuity follows immediately. (Details are left as exercise).

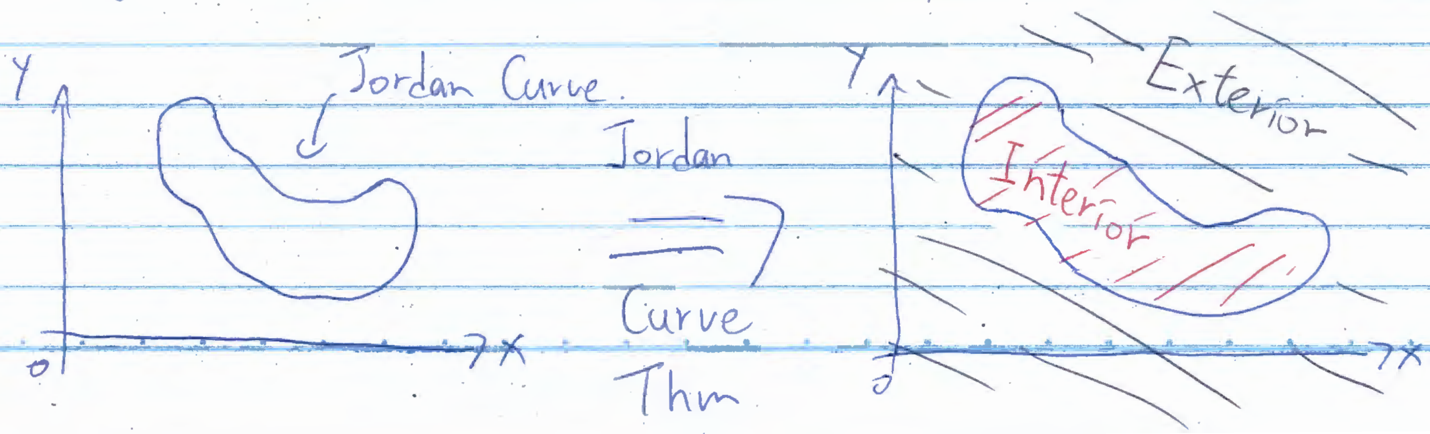
As a result, A_∞ is a curve with non-zero measure.
Such a curve is called <u>Osgood's curve</u>.

Now, we are going to construct a compact set D with non-zero measure boundary.
We need the following "obvious" theorem:

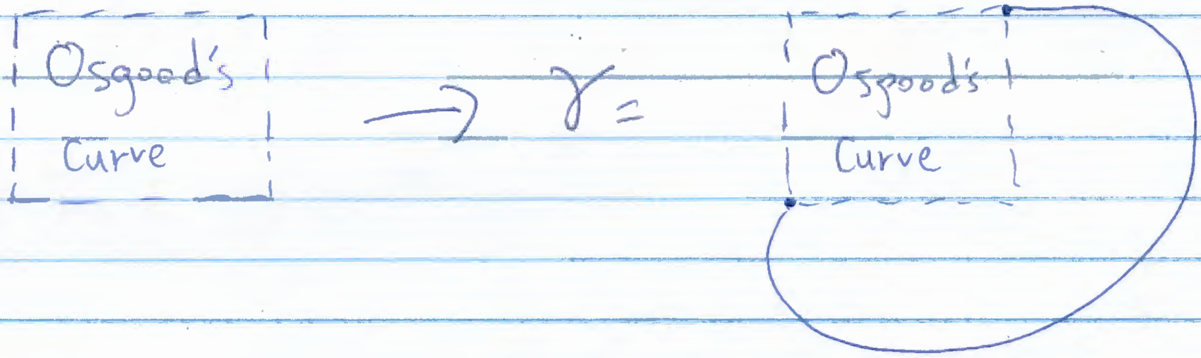
<u>Jordan Curve thm</u>:

- Let c be a Jordan curve.
  i.e. $c: [0,1] \to \mathbb{R}^2$ is injective and continuous, with $c(0) = c(1)$.
  Then the complement $\mathbb{R}^2 \setminus c$ consists of exactly two connected components. One of them is bounded (called <u>interior</u>) and one of them is unbounded (called <u>exterior</u>).
- Moreover, c is the boundary of both components.

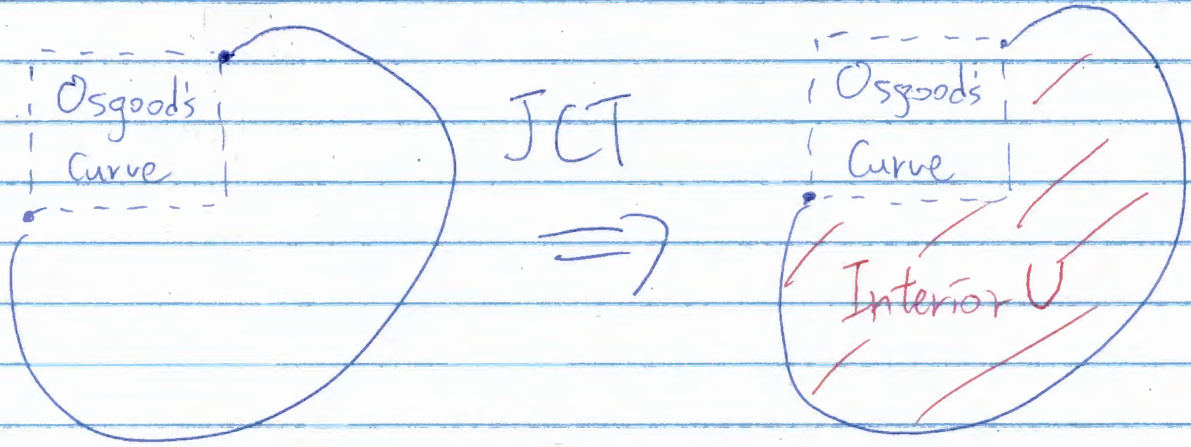
Forget all the mathematical stuff, it simply says the following:



From the discussion before, Osgood's curve is injective & continuous, so the curve γ below is a Jordan Curve:



By Jordan Curve thm, we get a bounded interior U with $\partial U = \gamma$:



Then $\bar{U} = U \cup \gamma$ is a compact set with non-zero measure boundary $\partial \bar{U} = \gamma$.

In other word, we need to assume that the boundary of compact set is measure zero in our course so that the integral always make sense.

Reference: ◦ The book "Space-Filling Curve" written by Hans Sagan (You may find it in library)
◦ Wikipedia.